Dr. rer.nat. Julian Koellermeier, Safiere Kuijpers, M.Sc.

Differential Equations In Science And Engineering | 23/24 IIa
Examination | 02.04.2024

Allowed Assistance:

- Pen, which is not writing red. No pencil.
- one double-sided, handwritten A4 paper sheet, including your name and student number.
- Additional assistance, calculators, mobile phones, are not allowed.

Hints:

- You have **120 minutes** for your examination. All answers have to be explained in detail.
- To pass the examination you need **55%** of the available points.
- Please start every exercise on the sheet, where the task is written on. If you are using additional sheets, note which exercise you refer to at the top of that sheet and write your name and student number on it.

Student number: ___ ___ ___ ___ ___ ___ ___

Name, First Name: ______________________________

Signature: ______________________________

Exercise	1	2	3	4	$\sum$
Points	5.0	5.0	5.0	5.0	20
Your Points					

Exercise 1.

We consider the following system (1) of chemical reactions for the four species A, B, C, D:

$$\begin{aligned} A + B &\xrightarrow{k_1} 2B \\ B + 2C &\xrightarrow{k_2} 3C \\ C + D &\xrightarrow{k_3} D \end{aligned} \tag{1}$$

a) Draw the reaction network.

b) Derive the stoichiometric net coefficients, the reaction rates, the production rates, and the corresponding system of ODEs that describes the dynamics of the species' concentrations denoted by n_A, n_B, n_C, n_D.

c) Show that the system has no conserved quantity that includes any of the species A, B, C.

d) We now want to change the last reactions of the system (1) such that the whole changed system has the conserved quantity $n_A + n_B + n_C + n_D$.

   Write down two examples for a changed last reaction.

   Show that in the changed systems $n_A + n_B + n_C + n_D$ is conserved.

0.5+2+1+1.5 points

Exercise 2.

The system of chemical reactions from exercise (1), can be simplified assuming constant populations n_A and n_D, and the scalings $x = \frac{n_B}{n_{B_0}}$, $z = \frac{n_C}{n_{C_0}}$. The simplified system reads

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x z^2, \\ \frac{dz}{dt} &= \beta x z^2 - \gamma z. \end{aligned}$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$ are some positive constants.

a) Compute the two steady states of this ODE system, including the non-trivial steady state $(x_s, z_s) = \left(\frac{\gamma}{\sqrt{\alpha\beta}}, \sqrt{\frac{\alpha}{\beta}}\right)$.

b) Find all possible parameters α, β, γ such that the Jacobian of the non-trivial steady state has two real eigenvalues with the same sign.

   Characterise the stability behavior of both steady states for that case.

c) We now interpret the ODE system as a model for the population dynamics of a prey species x and a predator species z.

   i) What is then the physical meaning of the parameters α, β, γ?

   ii) What is different with respect to the standard Lotka-Volterra model discussed in the lectures?

   iii) What process is modeled by that difference?

1+2.5+1.5 points

Exercise 3.

We consider the scalar wave equation

$$\frac{\partial^2}{\partial t^2}u - c\frac{\partial^2}{\partial x^2}u = 0, \quad c \in \mathbb{R}, \tag{2}$$

with constant wave velocity c.
We want to perform a linear stability analysis of equation (2) using the wave ansatz

$$u(t,x) = u_0 \cdot e^{i(kx-\omega t)}, \tag{3}$$

for wave number $k \in \mathbb{R}$, wave frequencies $\omega \in \mathbb{C}$ and amplitude $u_0 \in \mathbb{R}$.

a) What wave frequencies ω in (3) lead to a non-increasing, i.e. stable, wave in time?

b) Insert the wave ansatz (3) into the wave equation (2) to derive a stability condition for the wave equation. Show that the stability condition is equivalent to $c \geq 0$.

c) What is the physical interpretation of this stability condition and does it make sense?

1+3+1 points

Exercise 4.

The shallow water equations for water height $h(t,x)$ and vertical velocity $u(t,x)$ are

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \cos(\alpha) g \frac{h^2}{2} \end{pmatrix} = -\frac{1}{\lambda} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{4}$$

where $h(t,x)$ and $u(t,x)$ are the unknowns and g, λ, α are parameters.

a) What physical interpretations do the equations (4) have and what are the main assumptions for their derivation?

b) Show that the system (4) can be written in the following (so-called primitive variable) form:

$$\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} u & h \\ \cos(\alpha) g & u \end{pmatrix} \cdot \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = -\frac{1}{\lambda h} \begin{pmatrix} 0 \\ u \end{pmatrix}, \tag{5}$$

c) Assume the spatially homogeneous case in which all spatial derivatives vanish, i.e. $\partial_x h = 0$, $\partial_x u = 0$. For this case, derive the solution of the shallow water equations (4).

d) What problem can appear for numerical schemes trying to solve the homogeneous shallow water equations?

1.5+1.5+1.5+0.5 points

Name: Student number:

Name: Student number:

Name: Student number:

Name: Student number: